TIGHT FRAMES FOR EIGENSPACES OF THE LAPLACIAN ON DUAL POLAR GRAPHS

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Abstract. We consider $\Gamma = (X, E)$ a dual polar graph and we give a tight frame on each eigenspace of the Laplacian operator associated to Γ. We compute the constants associated to each tight frame and as an application we give a formula for the product in the Norton algebra attached to the eigenspace corresponding to the second largest eigenvalue of the Laplacian.

1. Introduction

In algebraic combinatorics a lot of research has been done on distance regular graphs. The main examples are the following families: Johnson, Grassmann, Hamming and dual polar graphs.

In this paper we consider the set of functions $I\!R^X = \{f : X \to I\!R\}$ where X is the set of vertices of the dual polar graphs. The distance on the graph gives rise to a Laplacian operator $\mathcal{L}$ on $I\!R^X$ and a decomposition of $I\!R^X$ into eigenspaces of $\mathcal{L}$. These topics can be seen in [5, 10, 11].

First we associate a lattice to the graph and characterize the eigenspaces of $\mathcal{L}$ in terms of this lattice. Instead of an orthogonal basis we can give a tight frame for each eigenspace. The theory of finite normalized tight frames has seen many developments and applications in recent years. See for instance the references in [3, 6, 7, 8, 9, 12, 13].

The eigenspace corresponding to the second largest eigenvalue of $\mathcal{L}$ is of particular importance since one can reconstruct the whole graph from the projections of the canonical basis onto it. We explicitly compute the constant of the tight frame attached to this eigenspace.

The notion of Norton algebra was developed to give realizations of the finite simple groups as automorphisms group of an algebra. The general construction starts with a graded algebra $\mathcal{V} = \bigoplus_i \mathcal{V}_i$ and gives an algebra structure on each subspace $\mathcal{V}_i$ by multiplying on $\mathcal{V}$ and then projecting onto $\mathcal{V}_i$.

As an application we answer a problem posed to us by Paul Terwilliger: a formula for the product in the Norton algebra attached to the eigenspace corresponding to the second largest eigenvalue of $\mathcal{L}$.

This article is organized as follows: In section 2 we give some classical definitions. In section 3, we associate a lattice to a dual polar graph Γ. In section 4, we give a convenient description for the eigenspaces V_i of $\mathcal{L}$. In the next section, Theorem 5.8 gives a tight frame on each eigenspace V_i and give a formula for the constant associated.

In the last section we compute an explicit formula for the product in the Norton algebra mentioned above.

2. Definitions

2.1. **Distance regular graphs.**

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Given $\Gamma = (X, E)$ a graph with distance $d(\ ,\ )$ we say that it is distance regular if for any $(x, y) \in X \times X$ such that $d(x, y) = h$ and for all $i, j \geq 0$ the cardinal of the set

$$\{z \in X \mid d(x, z) = i \text{ and } d(y, z) = j\}$$

is a constant denoted by p^h_{ij} which is independent of the pair (x, y).

2.2. **Adjacency algebra of a distance regular graph.**

Let $\Gamma = (X, E)$ be a distance regular graph of diameter d. Let $Mat_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with real entries, where the rows and columns are indexed by the elements of X.
For $0 \leq i \leq d$, let A_i denote the following matrix in $Mat_X(\mathbb{R})$:

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{if } d(x, y) \neq i \end{cases}$$

We call A_i the *ith adjacency matrix* of Γ. Using the definition it is not difficult to prove that the adjacency matrices of a distance regular graph satisfy:
(i') $A_0 = I$ where I is the identity matrix in $Mat_X(\mathbb{R})$;
(ii') $A_0 + \cdots + A_d = J$ where J is the all $1's$ matrix in $Mat_X(\mathbb{R})$;
(iii') $A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h$ $(0 \leq i, j \leq d)$.;
(iv') ${A_i}^t = A_i$
It follows from (i')-(iv') that $A_0, \ldots, A_d$ form a basis for a subalgebra $\mathcal{A}$ of $Mat_X(\mathbb{R})$. We call $\mathcal{A}$ the *adjacency algebra* of Γ.

It is known that the space of functions $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$ has a decomposition

$$\mathbb{R}^X = \oplus_{j=0}^{d} W_j$$

where $\{W_j\}_{j=0}^{d}$ are the common eigenspaces of $\{A_i\}_{i=0}^{d}$. Let $p_i(j)$ the eigenvalue of A_i on the eigenspace W_j.

By Proposition 1.1 of section 3.1 of Chapter III of [2], the adjacency matrices $\{A_i\}_{i=0}^{d}$ and the eigenvalues $\{p_i(j)\}_{i,j=0}^{d}$ of a given distance regular graph Γ satisfy:

$$A_i = v_i(A_1), \quad p_i(j) = v_i(\theta_j)$$

where $\theta_j = p_1(j)$, and $\{v_i\}_{i=0}^{d}$ are polynomials of degree i.

We will order the decomposition according to $\theta_0 > \theta_1 > ... > \theta_d$.

In Theorem 5.1 of III.5 of [2], one can find formulas for the polynomials associated to each Γ.

2.3. **Dual Polar Graphs.**

Let V be a finite dimensional vector space over $GF(q)$ (the finite field with q elements), together with a nondegenerate form ω. A subspace of V is called *isotropic* whenever the form vanishes completely on it. The dual polar graph corresponding to (V, ω) is the graph $\Gamma = (X, E)$ where

$$\begin{aligned} X &= \{v \subseteq V : \ v \text{ is maximal isotropic subspace}\} \\ E &= \{(u, v) \in X \times X \ : ,\ dim(u \cap v)) = d - 1\}. \end{aligned}$$

The dual polar graphs are distance regular, and are listed in page 274 of [1]. They are the following:

$$\begin{aligned}
C_d(q) : V &= GF(q)^{2d} \text{ with a nondegenerate sympletic form.}\\
B_d(q) : V &= GF(q)^{2d+1} \text{ with a nondegenerate quadratic form.}\\
D_d(q) : V &= GF(q)^{2d} \text{ with a nondegenerate quadratic form of Witt index } d.\\
{}^2D_{d+1}(q) : V &= GF(q)^{2d+2} \text{ with a nondegenerate quadratic form of Witt index } d.\\
{}^2A_{2d}(r) : V &= GF(r^2)^{2d+1} \text{ with a nondegenerate Hermitean form.}\\
{}^2A_{2d-1}(r) : V &= GF(r^2)^{2d} \text{ with a nondegenerate Hermitean form.}
\end{aligned}$$

In each of the cases above, the dimension of the maximal isotropic spaces is d.

We will denote $U^{\perp\!\!\!\perp} = \{v \in V : \omega(v, u) = 0 \ \forall \ u \in U\}$.

In each case there is a group acting on these spaces, namely the group G_ω of linear transformations on the underlying space V that preserve the form ω.

3. Lattice associated with dual polar graphs

In this section we consider the graphs defined above and we associate a lattice to them. We recall the following definitions:

- A partial order is a binary relation "$\leq$" over a set P which is reflexive, antisymmetric, and transitive.
- A partially ordered set (POSET) $(P, \leq)$ is a set P with a partial order $\leq$.
- A lattice $(\mathbf{L}, \leq, \wedge, \vee)$ is a POSET $(\mathbf{L}, \leq)$ in which every pair of elements $u, w \in \mathbf{L}$ has a least upper bound and a greatest lower bound. The first is called the join and it is denoted by $u \vee w$ and the second is called the meet and it is denoted by $u \wedge w$.

3.1. **Construction of the lattice.**

Let $\Gamma = (X, E)$ be a dual polar graph and be V be the underlying finite dimensional vector space over $GF(q)$.

Definition 3.1.

$$\begin{aligned}
\Omega_\ell &= \{v \subseteq V : \ v \textit{ is an isotropic subspace and } dim(v) = \ell\}.\ \ell = 0, ...d.\\
\Omega_{d+1} &= \{V\}
\end{aligned}$$

We let $\hat{0} := \{0\}$ *and* $\hat{1} := V$ *and we denote* Ω_d *by* X*.*

We will always work with $d > 1$, i.e., $\Omega_1 \neq X$.

Definition 3.2.

- $\mathbf{L} = \cup_{\ell=0}^{d+1} \Omega_\ell$.
- *Given isotropic subspaces* $u, w \subseteq V$ *we set:*
  - $u \leq w$ *if and only if,* u *is a subspace of* w*.*
  - $u \wedge w = u \cap w$
  - $u \vee w = span\{u, w\}$ *if that space is isotropic, otherwise,* $u \vee w = V = \hat{1}$
- *The rank of* $w \in \Omega_\ell$ *is* ℓ *and it is denoted by* $\mathrm{rk}(w)$*.*
- *Given* $w \in \Omega_j$*;* u *covers* w *or* w *is covered by* u*, if* $u \in \Omega_{j+1}$ *and* $w \leq u$*. We denote it by* $u \gtrdot w$ *or* $w \lessdot u$*.*
- *An atom is an element that covers* $\hat{0}$ *and a coatom is an element covered by* $\hat{1}$*.*

Is not difficult to see that $(\mathbf{L}, \leq, \wedge, \vee)$ is a finite lattice with lowest element $\hat{0}$ and greatest element $\hat{1}$. In our notation, the set of atoms is Ω_1 and the set of coatoms is X.

Lemma 3.3. *The lattice* **L** *has the following properties:*

(1) **L** *is atomic.*
(2) $u \vee w \neq \hat{1} \Rightarrow \mathrm{rk}(u) + \mathrm{rk}(w) = \mathrm{rk}(u \vee w) + \mathrm{rk}(u \wedge w)$

Proof.

(1) Each element $u \in \Omega_j$ of the lattice is a subspace of $GF(q)^n$, so taking a basis $\{v_1, ..., v_j\}$ of u, we obtain that $u = span(v_1) \vee span(v_2) \vee ... \vee span(v_j)$ is a join of atoms.
(2) The rank of an element is the dimension, so the formula is true because of the well known identity $dim(u + w) = dim(u) + dim(w) - dim(u \cap w)$. (The formula fails for the case $u \vee w = \hat{1}$ because then $u \vee w$ is not equal to $u + w$).

QED.

Corollary 3.4. *If τ and σ are different atoms such that $\tau \vee \sigma \neq \hat{1}$, then*

$$\mathrm{rk}(\tau \vee \sigma) = 2.$$

Proof.

$$\begin{aligned} \mathrm{rk}(\tau \vee \sigma) &= \mathrm{rk}(\tau) + \mathrm{rk}(\sigma) - \mathrm{rk}(\tau \wedge \sigma) \\ &= 1 + 1 - 0 \\ &= 2 \end{aligned}$$

QED.

Lemma 3.5. *Let u and w be elements of the lattice which are not coatoms. If $u \vee w$ covers both u and w then u and w both cover $u \wedge w$.*

Reciprocally if u and w cover $u \wedge w$ and $u \vee w \neq \hat{1}$, then $u \vee w$ covers both u and w.

Proof. In order to prove the first statement, observe that z covers w iff $z \geq w$ and $\mathrm{rk}(z) = \mathrm{rk}(w) + 1$. So, $u \vee w$ covers both u and w iff $\mathrm{rk}(u \vee w) = \mathrm{rk}(u) + 1 = \mathrm{rk}(w) + 1$ (in particular, we must have that $\mathrm{rk}(u) = \mathrm{rk}(w)$). Also, since u and w are not coatoms and $\mathrm{rk}(u \vee w) = \mathrm{rk}(u) + 1$ we deduce that $u \vee w \neq \hat{1}$. Then, by Lemma 3.3 (2),we get $\mathrm{rk}(u) + \mathrm{rk}(w) - \mathrm{rk}(u \wedge w) = \mathrm{rk}(u) + 1$, i.e., $\mathrm{rk}(w) = \mathrm{rk}(u \wedge w) + 1$, which implies that w covers $u \wedge w$. The proof is similar for u.

Reciprocally, if u and w cover $u \wedge w$, then $\mathrm{rk}(u) = \mathrm{rk}(w) = \mathrm{rk}(u \wedge w) + 1$. Using Lemma 3.3 (2) we get $\mathrm{rk}(w) = \mathrm{rk}(u) + \mathrm{rk}(w) - \mathrm{rk}(u \vee w) + 1$ which implies $\mathrm{rk}(u) + 1 = \mathrm{rk}(u \vee w)$ and then that $u \vee w$ covers u (and similarly w). QED.

4. Description of the eigenspaces of a dual polar graph using the associated lattice

In this section we will consider a dual polar graph $\Gamma = (X, E)$, together with its associated decomposition:

$$I\!R^X = \oplus_{i=0}^d W_i,$$

where $\{W_i\}_{i=0}^d$ are the common eigenspaces of the adjacency matrices of Γ.

We will describe each of the eigenspaces $\{W_i\}_{i=0}^d$, using the lattice previously defined.

For ease of writing, we will use the following notation:

Notation 4.1.

- *For any statement P, let* $[P] = \begin{cases} 1 & \textit{if } P \textit{ is true.} \\ 0 & \textit{if } P \textit{ is false.} \end{cases}$
- $I\!R^X = \{f : X \to I\!R\}$,
  (thus $0 \in I\!R^X$ *will denote the function such that* $0(x) \equiv 0, \forall x \in X$*,*
  analogously $1 \in I\!R^X$*.)*
- *Let* $<;>$ *be the inner product in* $I\!R^X$ *defined by* $< f, g >= \sum_{x \in X} f(x)g(x)$.
- *For* $u \subseteq I\!R^X$ *let* $u^{\perp} = \{f \in I\!R^X : < f; g >= 0 \ \forall\ g\ \in U\}$
- $||f||^2 =< f; f >$.

We will need the following lemma.

Recall that $\begin{bmatrix} i \\ 1 \end{bmatrix}_q = \begin{cases} \frac{q^i - 1}{q-1} & \forall\ i \geq 1 \\ 0 & \forall\ i < 1 \end{cases}$ and $\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{\begin{bmatrix} i \\ 1 \end{bmatrix}_q \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \cdots \begin{bmatrix} i-j+1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} j \\ 1 \end{bmatrix}_q \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q}$.

Lemma 4.2. *(9.4.2 of* [1]*)*
Let e be $1, 1, 0, 2, \frac{3}{2}, \frac{1}{2}$ in the respective cases:
$C_d(q), B_d(q), D_d(q),^2 D_{d+1}(q),^2 A_{2d}(r),^2 A_{2d-1}(r)$.
Let W be a fixed isotropic space of dimension j. The number of isotropic spaces U of dimension $(k + l + m)$, meeting W in a space of dimension m and $W^{\perp}$ in space of dimension $l + m$ is:

$$q^{l(j-m)+k(2d-j-m-2l+e-1)-k(k-1)/2} \begin{bmatrix} j \\ m \end{bmatrix}_q \begin{bmatrix} j-m \\ k \end{bmatrix}_q \begin{bmatrix} d-j \\ l \end{bmatrix}_q \prod_{i=0}^{l-1} (1 + q^{d+e-j-i-1})$$

Lemma 4.3. *For $z \in \Omega_j$, let $a_j = |\{x \in X : z \leq x\}|$. Then*

$$a_j = \prod_{i=0}^{d-j-1} (1 + q^{e+i}) \quad \text{if } 0 \leq j \leq d \quad \text{and} \quad a_{d+1} = 0$$

Proof. We use the previous lemma with $W = z \in \Omega_j$ and $U = x \in \Omega_d$. Then $k + l + m = d$, $m = j$ and $l + m = d$. Therefore we can apply the formula with $k = 0,\ l = d - j$. QED.

Remark 4.4. *Note that* $a_j = (1 + q^{d-j-1+e})\ a_{j+1}$.

Definition 4.5.
$\iota : \mathbf{L} \to I\!R^X$ *is the map defined by* $\iota(z)(x) = [z \leq x]\ \forall\ z \in \mathbf{L},\ x \in X$

Lemma 4.6.

i) $\iota(\hat{1}) = 0\ \in\ I\!R^X$ *ii)* $\iota(\hat{0}) = 1\ \in\ I\!R^X$ *iii)* $\iota(z)\iota(y) = \iota(z \vee y)\ \forall\ z, y \in \mathbf{L}$.

Proof.
$i)\iota(\hat{1})(x) = [\hat{1} \leq x] = 0\ \forall x$, since $\hat{1}$ is above the x's.
$ii)\iota(\hat{0})(x) = [\hat{0} \leq x] = 1\ \forall x$.
$iii)$

$$\begin{aligned} \iota(z)(x)\iota(y)(x) &= [z \leq x][y \leq x] \\ &= [(z \leq x) \textit{ and } (y \leq x)] \\ &= [z \vee y \leq x] \\ &= \iota(z \vee y)(x) \end{aligned}$$

QED.

Lemma 4.7.

$$< \iota(z); \iota(y) >= ||\iota(z \vee y)||^2 \quad \forall\ z, y \in \mathbf{L}.$$

Proof.

$$\begin{aligned} <\iota(z);\iota(y)> &= \sum_{x\in X}\iota(z)(x)\iota(y)(x) \\ &= \sum_{x\in X}\iota(z\vee y)(x) \quad \text{(by Lemma 4.6 iii))} \\ &= \sum_{x\in X}(\iota(z\vee y)(x))^2 \quad (\text{since } \iota(z\vee y)(x)\in\{0,1\}) \\ &= ||\iota(z\vee y)||^2 \end{aligned}$$

QED.

Corollary 4.8. *$z\vee y=\hat{1}$ if and only if $\iota(z)$ and $\iota(y)$ are orthogonal to each other.*

Proof. By lemmas 4.7 and 4.6 i), $<\iota(z);\iota(y)>=0$ if and only if $z\vee y=\hat{1}$. QED.

Lemma 4.9.

$$||\iota(z)||^2=a_j \quad \forall\ z\in\Omega_j,\ j=0,1,...,d$$

Proof.

$$\begin{aligned} ||\iota(z)||^2 &= |\{x\in X:\iota(z)(x)=1\}| \\ &= |\{x\in X: z\le x\}| \\ &= a_j \end{aligned}$$

QED.

Corollary 4.10. *If $z\vee y\in\Omega_j$, then $<\iota(z);\iota(y)>=a_j$*

Proof. Direct from the two previous lemmas. QED.

Lemma 4.11. *If τ and σ are both atoms then:*

$$<\iota(\tau);\iota(\sigma)>=\begin{cases} a_1 & \text{if } \tau=\sigma \\ 0 & \text{if } \tau\vee\sigma=\hat{1} \\ a_2 & \text{otherwise} \end{cases}$$

Proof. If $\tau=\sigma$, then $\tau\vee\sigma=\tau$, and so we have that $<\iota(\tau);\iota(\sigma)>=a_1$ by Lemma 4.9. If $\tau\vee\sigma=\hat{1}$, then by Lemma 4.7, we have that $<\iota(\tau);\iota(\sigma)>=0$.

If $\tau\neq\sigma$ and $\tau\vee\sigma\neq\hat{1}$, then by Lemma 3.4, $\tau\vee\sigma\in\Omega_2$, so by Corollary 4.10, we have $<\iota(\tau);\iota(\sigma)>=a_2$. QED.

4.1. **A filtration for $\mathbb{R}^X$.**

Definition 4.12.
For $j=0,1,...,d$, let $\Lambda_j\subseteq\mathbb{R}^X$ be the subspace generated by $\{\iota(x)\}_{x\in\Omega_j}$, that is $\Lambda_j=span(\iota(\Omega_j))$.

We want to show that $\Lambda_j\subseteq\Lambda_{j+1}$. For this we need some tools first.

Definition 4.13. *Given $w\in\mathbf{L}$, let:*

$$w^*=\sum_{v\gtrdot w}\iota(v)$$

Lemma 4.14. *Given $w\in\mathbf{L}$, $\iota(w)$ is a scalar multiple of w^*. In fact,*

$$w^*=\begin{bmatrix} d-j \\ 1\end{bmatrix}_q\iota(w) \quad \forall\ w\in\Omega_j$$

Proof. Given $x \in X$, we have:

$$\begin{aligned} w^*(x) &= \sum_{v \gtrdot w} [v \leq x] \\ &= |\{v : w \lessdot v \leq x\}| \quad (*) \end{aligned}$$

Clearly, if $w \not\leq x$, that number is zero, i.e., $w^*(x) = 0 = \iota(w)(x)$ if $w \not\leq x$. On the other hand, if $w \leq x$, the number in $(*)$ is the number of spaces in x built from w by adding a one-dimensional space. That one-dimensional space must be in x and not in w, so there are $\left[\begin{smallmatrix} d-j \\ 1 \end{smallmatrix}\right]_q$ ways of doing this. So

$$w^*(x) = \begin{cases} 0 & \text{if} \quad w \not\leq x \\ \left[\begin{smallmatrix} d-j \\ 1 \end{smallmatrix}\right]_q & \text{if} \quad w \leq x, \quad \text{then} \end{cases}$$

$$w^* = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \iota(w).$$

QED.

Corollary 4.15.

$$\Lambda_0 \subseteq \Lambda_1 \subseteq ... \subseteq \Lambda_d = I\!R^X$$

Proof. Let $f \in \Lambda_j$. We can assume that $f = \iota(w)$. By definition, $w^* \in \Lambda_{j+1}$. But by lemma 4.14, $\iota(w)$ is a non-zero scalar multiple of w^*, so $f \in \Lambda_{j+1}$

QED.

Definition 4.16. *Let $V_0 = \Lambda_0$ and $V_j = \Lambda_j \cap \Lambda_{j-1}^{\perp} \quad j = 1, ..., d$.*

So, we have that $\Lambda_j = V_0 \oplus V_1 \oplus ... \oplus V_j$.

We want to show that for $j = 1, ..., d$, $V_j \neq \{0\}$, that is $\Lambda_{j-1} \neq \Lambda_j$. To prove this, we need more lemmas.

Definition 4.17. *Let $\mathcal{L} : I\!R^X \mapsto I\!R^X$ denote the Laplacian operator defined by*

$$\mathcal{L}(f)(x) = \sum_{y \in X:\ d(x,y)=1} f(y)$$

Observe that

$$\begin{aligned} \mathcal{L}(f)(x) &= \sum_{y \in X} [d(x,y) = 1] f(y) \\ &= \sum_{y \in X} (A_1)_{xy} f(y), \end{aligned}$$

where A_1 is the first adjacency matrix of $\Gamma = (X, E)$ a dual polar graph. So $\mathcal{L}$ can be thought as multiplication by A_1. In particular, $\mathcal{L}$ is symmetric and $< \mathcal{L}(f), g >=< f, \mathcal{L}(g) >$.

Lemma 4.18. *If $x \in X$, then $\mathcal{L}(\iota(x)) = \sum_{y \in X:\ d(x,y)=1} \iota(y)$.*

Proof.

$$\begin{aligned}
\mathcal{L}(\iota(x))(z) &= \sum_{y\in X:\ d(z,y)=1} \iota(x)(y) \\
&= \sum_{y\in X}[d(z,y)=1][x\leq y] \\
&= \sum_{y\in X}[d(z,y)=1][x=y] \qquad \text{because } x\leq y \iff x=y \text{ since } x,y\in X \\
&= [d(z,x)=1] \quad \text{while}
\end{aligned}$$

$$\begin{aligned}
\Big(\sum_{y\in X:\ d(x,y)=1} \iota(y)\Big)(z) &= \sum_{y\in X}[d(x,y)=1][y\leq z] \\
&= \sum_{y\in X}[d(x,y)=1][y=z] \\
&= [d(x,z)=1]
\end{aligned}$$

QED.

Lemma 4.19. *Let $x\in X$. Then:*

$$\mathcal{L}(\iota(x)) = -\begin{bmatrix} d \\ 1 \end{bmatrix}_q \iota(x) + \sum_{z:z\lessdot x}\iota(z)$$

Proof.

$$\begin{aligned}
\Big(\sum_{z:z\lessdot x}\iota(z)\Big)(y) &= \sum_{z:z\lessdot x}[z\leq y] \\
&= \sum_{z\in\Omega_{d-1}}[z\leq x\wedge y] \\
&= \begin{cases} \sum_{z\in\Omega_{d-1}}[z\leq x] & if x=y \\ \sum_{z\in\Omega_{d-1}}[z=x\wedge y] & if x\neq y \end{cases} \\
&= \begin{cases} \left[\begin{smallmatrix} d \\ 1 \end{smallmatrix}\right]_q & if x=y \\ [x\wedge y\in\Omega_{d-1}] & if x\neq y \end{cases} \\
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q [x=y] + [x\wedge y\in\Omega_{d-1}][x\neq y] \\
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q [x=y] + [d(x,y)=1] \\
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \iota(x)(y) + \mathcal{L}(\iota(x))(y)
\end{aligned}$$

QED.

Proposition 4.20.
Let $\Gamma=(X,E)$ be a dual polar graph and let e be as in Lemma 4.2.
For $j=0,1,...,d-1$ and for all $w\in\Omega_j$ the following holds:

$$\sum_{u:u\gtrdot w}\sum_{z:z\lessdot u}\iota(z) = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left((q^j+q^{d+e-j-1})\iota(w) + \sum_{v:v\lessdot w}\iota(v) \right)$$

Proof. Let us call $S(x)$ the function on the left hand side of the equation above and $R(x)$ the function on the right hand side.

We have to see that evaluating on an arbitrary $x \in X$, they are both equal.

(1) **CASE 1**: $w \wedge x \in \Omega_j$.

In this case, $w \wedge x = w$ that is $w \leq x$. Then

$$R(x) = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left((q^j + q^{d+e-j-1}) + |\{v :\ v \lessdot w \text{ and } v \leq x\}|\right)$$

However, $v \lessdot w \leq x \Rightarrow v \leq x$, so

$$R(x) = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left((q^j + q^{d+e-j-1}) + |\{v :\ v \lessdot w\}|\right)$$

Since $|\{v :\ v \lessdot w\}|$ is the number of spaces of dimension $j-1$ in a space of dimension j, i.e., $\left[\begin{smallmatrix} j \\ j-1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]_q$, we conclude:

$$R(x) \quad = \quad \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left((q^j + q^{d+e-j-1}) + \begin{bmatrix} j \\ 1 \end{bmatrix}_q\right)$$

On the other hand, $S(x) = \sum_{u \gtrdot w} |\{z :\ z \lessdot u \text{ and } z \leq x\}|$(recall that $w, z \in \Omega_j,\ u \in \Omega_{j+1}$ and $x \in \Omega_d$).

If $u \leq x$ then $z \lessdot u \Rightarrow z \leq x$, i.e.:

$$|\{z :\ z \lessdot u \text{ and } z \leq x\}| = |\{z :\ z \lessdot u\}| = \begin{bmatrix} j+1 \\ j \end{bmatrix}_q = \begin{bmatrix} j+1 \\ 1 \end{bmatrix}_q$$

If $u \not\leq x$, then $x \wedge u \neq u$. But $w \leq x \wedge u \leq u$ and $w \lessdot u$, so $x \wedge u = w$. So in this case $\{z :\ z \lessdot u : z \leq x\} = \{w\}$ and thus, $|\{z :\ z \lessdot u : z \leq x\}| = 1$. Therefore,

$$\begin{aligned} S(x) \quad &= \quad |\{u :\ w \lessdot u \leq x\}| \begin{bmatrix} j+1 \\ 1 \end{bmatrix}_q + |\{u :\ u \gtrdot w \text{ and } u \not\leq x\}| \\ &= \quad \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \begin{bmatrix} j+1 \\ 1 \end{bmatrix}_q + |\{u :\ u \gtrdot w \text{ and } u \not\leq x\}| \end{aligned}$$

This last number can be computed from

$$|\{u : u \gtrdot w \text{ and } u \not\leq x\}| = |\{u :\ u \gtrdot w\}| - |\{u :\ x \geq u \gtrdot w\}|$$

For this, we need Lemma 4.2.

In order to compute $|\{u :\ u \gtrdot w\}|$ we fix w isotropic of dimension j and we want to compute the number of isotropic spaces u of dimension $j+1$. Since $u \geq w$ we have $dim(u \cap w) = dim\ w = j$ and since u must be isotropic, $u \subseteq w^{\perp\!\!\!\perp}$ and then $dim(u \cap w^{\perp\!\!\!\perp}) = dim\ u = j+1$. Then in the notation of the lemma "W"$= w$, "U"$= u$ and

$$k + l + m = j + 1, \quad m = j, \quad \text{and} \quad l + m = j + 1$$

Therefore $l = 1$, $k = 0$.

Then, the lemma gives us

$$\begin{aligned}
|\{u : u \gtrdot w\}| &= \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q (1+q^{d+e-j-1}) \\
|\{u : u \gtrdot w \ , \ u \not\leq x\}| &= |\{u \gtrdot w\}| - |\{u : x \geq u \gtrdot w\}| \\
&= \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q (1+q^{d+e-j-1}) - \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q q^{d+e-j-1} \\
\text{and } S(x) &= \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} j+1 \\ 1 \end{bmatrix}_q + q^{d+e-j-1} \right) \\
\Rightarrow S(x) &= R(x)
\end{aligned}$$

(2) **CASE 2**: $w \wedge x \in \Omega_k$, for $k < j-1$.

In this case, $w \not\leq x$ and $w \wedge x \not\lessdot w$. Then $S(x)$ is the number of pairs (u, z) which satisfy:

(a) $w \lessdot u$
(b) $z \lessdot u$
(c) $z \leq x$

(Recall that $w, z \in \Omega_j,\ u \in \Omega_{j+1}$ and $x \in \Omega_d$).

Now, $((c) + w \not\leq x) \Rightarrow z \neq w$. This plus (a) and (b) implies $w \vee z = u$, i.e., $w, z \lessdot w \vee z$. Then, $w \wedge z \lessdot w, z$ (Lemma 3.5).
However, $(c) \Rightarrow w \wedge z \leq w \wedge x$ $(\leq w)$, so since $w \wedge z \lessdot w$, we have only two options for $w \wedge x$:

- $w \wedge x = w \wedge z$ (impossible since $w \wedge x \not\lessdot w$ while $w \wedge z \lessdot w$)
- $w \wedge x = w$, which would imply $w \leq x$.

Since neither of these happens in CASE 2, we conclude that there are no such pairs (u, z), and $S(x) = 0$.

On the other hand,

$$R(x) = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \left((q^j + q^{d+e-j-1})\ 0 + |\{v :\ v \lessdot w \text{ and } v \leq x\}| \right)$$

But $(v \lessdot w$ and $v \leq x) \Rightarrow v \leq w \wedge x \leq w$.

Again, $w \wedge x \neq w$, because $w \not\leq x$ and $w \wedge x \neq v$ because $v \lessdot w$ while $w \wedge x \not\lessdot w$. So $R(x) = 0$ too and we have equality in this case.

(3) **CASE 3**: $w \wedge x \in \Omega_{j-1}$

In this case $w \not\leq x$ and $w \wedge x \lessdot w$.

As in CASE 2, $R(x) = \left[\begin{smallmatrix} d-j \\ 1 \end{smallmatrix}\right]_q |\{v :\ v \lessdot w \text{ and } v \leq x\}|$, and given such a v, we deduce that either $w \wedge x = w$ (impossible since $w \wedge x \in \Omega_{j-1}$) or $w \wedge x = v$. Then $\{v :\ v \lessdot w \text{ and } v \leq x\} = \{w \wedge x\}$ so $R(x) = \left[\begin{smallmatrix} d-j \\ 1 \end{smallmatrix}\right]_q$.
On the other hand

$$S(x) = \sum_{u:\ u \gtrdot w} \sum_{z} [z \lessdot u][z \leq x] = \sum_{u:\ u \gtrdot w} \sum_{z} [z \leq u \wedge x][z \lessdot u]$$

But $z \leq u \wedge x \leq u$ and $z \lessdot u$ imply that $u \wedge x$ is either z or u.
If $u \wedge x = u$, $\Rightarrow u \leq x$, and since $w \lessdot u$, then $w \leq x$, which can not happen in CASE 3. Therefore, $u \wedge x = z$, i.e.,

$$\sum_{z} [z \leq u \wedge x][z \lessdot u] = \begin{cases} 1 & \text{if } u \wedge x \lessdot u \\ 0 & \text{otherwise} \end{cases}$$

So assume $u \wedge x \lessdot u$. Since $u \wedge x \leq x$ and $w \not\leq x$, we must have $u \wedge x \neq w$. Since $u \gtrdot w$ and $u \gtrdot u \wedge x$, then $u = (u \wedge x) \vee w$, i.e., u is determined by

$z = u \wedge x$, i.e.,

$$S(x) = |\{z : \ z \leq x \text{ and } z \vee w \gtrdot w, z\}|.$$

Since $z \vee w \gtrdot w, z \Rightarrow w \wedge z \lessdot w$ and $w \wedge z \leq w \wedge x \lessdot w$, we conclude that $w \wedge z = w \wedge x$. Also, in order to have $z \vee w \gtrdot w$, i.e., to be isotropic, we must have $z \leq w^{\perp\!\!\!\perp}$. Hence, $S(x) = |\{z : \ w \wedge x \lessdot z \leq w^{\perp\!\!\!\perp} \cap x\}| = \begin{bmatrix} \theta(x) \\ 1 \end{bmatrix}_q$, where $\theta(x) = dim(w^{\perp\!\!\!\perp} \cap x) - dim(w \wedge x) = dim(w^{\perp\!\!\!\perp} \cap x) - (j-1)$.

Since $w \wedge x \lessdot w$, we have that $w = (w \wedge x) \vee \tau$, for some atom $\tau \nleq x$. Since $w^{\perp\!\!\!\perp} \cap x = \{\alpha \in x : \ \omega(\alpha, \beta) = 0 \quad \forall\ \beta \in w\}$ and for all $\alpha \in x$, $\omega(\alpha, \beta) = 0$ for all $\beta \in w \wedge x$ (because x is isotropic), then we have that

$$w^{\perp\!\!\!\perp} \cap x = \{\alpha \in x : \ \omega(\alpha, \tau) = 0\} = \tau^{\perp\!\!\!\perp} \cap x.$$

Letting V be the underlying space of definition 2.3 we have:

$$\begin{aligned} dim(w^{\perp\!\!\!\perp} \cap x) &= dim(\tau^{\perp\!\!\!\perp} \cap x) \\ &= dim(\tau^{\perp\!\!\!\perp}) + dim(x) - dim(span\{\tau^{\perp\!\!\!\perp}, x\}) \\ &= dimV - 1 + d - dim(span\{\tau^{\perp\!\!\!\perp}, x\}) \end{aligned}$$

But $dim(span\{\tau^{\perp\!\!\!\perp}, x\})$ can be either $dim(\tau^{\perp\!\!\!\perp}) = dim(V) - 1$, or else is $dim(V)$. In the first case, we would have that $x \subset \tau^{\perp\!\!\!\perp}$, which would imply that $span\{x, \tau\}$ is isotropic, absurd since x is maximal isotropic and $\tau \nleq x$. Therefore, we have the second case, and $dim(w^{\perp\!\!\!\perp} \cap x) = (dimV) - 1 + d - dim(V) = d - 1$. This implies that $\theta(x) = (d-1) - (j-1) = d - j$ and $S(x) = \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q = R(x)$.

QED.

Lemma 4.21. *Consider the same hypothesis of the previous Proposition and let Λ_j be defined as in 4.12.*

Then, there are constants $\mu_0 > \mu_1 > ... > \mu_d$ such that for every $v \in \Lambda_j$, there exists $v' \in \Lambda_{j-1}$ with $\mathcal{L}(v) = \mu_j v + v'$.

Proof. It is enough to show the lemma for each element of the spanning set $\{\iota(x)\}_{x \in \Omega_j}$. We prove it inductively starting at $j = d$.

Taking $\iota(x) \in \Lambda_d$, by Lemma 4.19, we have that $\mathcal{L}(\iota(x)) = -\begin{bmatrix} d \\ 1 \end{bmatrix}_q \iota(x) + \sum_{z \lessdot x} \iota(z)$. Since $\sum_{z \lessdot x} \iota(z) \in \Lambda_{d-1}$ the proposition holds for $j = d$ with $\mu_d = -\begin{bmatrix} d \\ 1 \end{bmatrix}_q$.

Assume now that the following inductive hypothesis is true for $j + 1$:

$$\textit{There are constants } \mu_{j+1} \textit{ such that } \mathcal{L}(\iota(u)) = \mu_{j+1}\iota(u) + \sum_{z \lessdot u} \iota(z) \ \forall\ u \in \Omega_{j+1}$$

Let $\iota(w) \in \Lambda_j$. Let's recall that by Lemma 4.14 $\iota(w) = \frac{1}{\begin{bmatrix} d-j \\ 1 \end{bmatrix}_q} \sum_{u \gtrdot w} \iota(u)$.

$$\begin{aligned}
\mathcal{L}(\iota(w)) &= \frac{1}{\left[{d-j \atop 1}\right]_q} \sum_{u>w} \mathcal{L}(\iota(u)) \\
&= \frac{1}{\left[{d-j \atop 1}\right]_q} \sum_{u>w} \Big(\mu_{j+1}\iota(u) + \sum_{z<u} \iota(z)\Big) \\
&= \mu_{j+1} \frac{1}{\left[{d-j \atop 1}\right]_q} \sum_{u>w} \iota(u) + \frac{1}{\left[{d-j \atop 1}\right]_q} \sum_{u>w}\sum_{z<u} \iota(z) \\
&= \mu_{j+1}\iota(w) + (q^j + q^{d+e-j-1})\iota(w) + \sum_{v<w} \iota(v) \\
&= \left(\mu_{j+1} + q^j + q^{d+e-j-1}\right)\iota(w) + \sum_{v<w} \iota(v)
\end{aligned}$$

This proves the inductive hypothesis by taking $\mu_j = \mu_{j+1} + (q^j + q^{d+e-j-1})$
In particular we have $\mu_j > \mu_{j+1}$ and the lemma is proved. QED.

Corollary 4.22. *For $j = 0, ..., d$, Λ_j are $\mathcal{L}$-invariant subspaces of $I\!R^X$.*

Proof. This follows directly by the previous lemma and Corollary 4.15. QED.

Theorem 4.23.
For $j = 0, ..., d$, $V_j = \Lambda_j \cap \Lambda_{j-1}^{\perp}$, are eigenspaces of $\mathcal{L}$ with corresponding eigenvalue $\mu_j = q^e \left[{d-j \atop 1}\right]_q - \left[{j \atop 1}\right]_q$.

Proof. Take $v \in V_j$ $(\subseteq \Lambda_j)$. By Lemma 4.21 $\mathcal{L}(v) = \mu_j v + v'$ with $v' \in \Lambda_{j-1}$ and by Corollary 4.22 $\mathcal{L}(v') \in \Lambda_{j-1}$. Then by definition of V_j:

$$\begin{aligned}
0 &= < v, \mathcal{L}(v') > \\
&= < \mathcal{L}(v), v' > \\
&= < \mu_j v + v', v' > \\
&= < \mu_j v, v' > + < v', v' > \\
&= ||v'||^2
\end{aligned}$$

thus $\mathcal{L}(v) = \mu_j v \; \forall \; v \in V_j$. Therefore, $I\!R^X = \bigoplus_{j=0}^d V_j$ where each V_j is either zero or an eigenspace of $\mathcal{L}$.

Since $X = \Omega_d$ is the set of vertices of a dual polar graph $\Gamma = (X, E)$ (a distance regular graph of diameter d), recall that there are exactly $d + 1$ eigenspaces of the adjacency matrix A_1, therefore of $\mathcal{L}$. Thus each V_j is indeed an eigenspace of $\mathcal{L}$ (hence $V_j \neq 0 \; \forall \; j$) and μ_j are the eigenvalues of $\mathcal{L}$. Their values can be obtained by induction from the formula $\mu_j = \mu_{j+1} + (q^j + q^{d+e-j-1})$, or directly from Theorem 9.4.3 of [1] (since $\mu_j > \mu_{j+1}$).

QED.

5. Tight Frames for the eigenspaces of a dual polar graph

In this section we will consider $\Gamma = (X, E)$ be a dual polar graph of diameter d, **L** the associated lattice described in Section 3 and

$$I\!R^X = \oplus_{j=0}^d V_j$$

the corresponding decomposition.

We will give a finite tight frame on each V_j and a formula for the corresponding constants. We will compute explicitely the constant associated to the eigenspace of the second largest eigenvalue.

Definition 5.1.

Given a vector space $(V, <;>)$ *a finite tight frame on* V *is a finite set* $F \subseteq V$ *which satisfies the following condition: there exists a non-zero constant* λ *such that:*

$$\sum_{v \in F} | < f, v > |^2 = \lambda \parallel f \|^2 \quad \forall\, f \in V$$

As a consequence, f can be expanded as follows: $f = \frac{1}{\lambda} \sum_{v \in F} < f, v > v$

Definition 5.2. *For* $j = 0, 1, ..., d$*, let* $U^j \in I\!R^{X \times X}$ *be the matrix*

$$(U^j)_{x,y} = (x, y)^j \quad \textit{where} \quad (x, y)^j = \sum_{u \in \Omega_j} \iota(u)(x)\iota(u)(y)$$

Lemma 5.3. *For* $j = 0, 1, ..., d$

$$U^j = \sum_{l=j}^{d} \begin{bmatrix} l \\ j \end{bmatrix}_q A_{d-l}$$

where A_i *is the* i*-th adjacency matrix of the dual polar graph.*

Proof. Let $(x, y) \in X \times X$ and $l = \mathrm{rk}(x \wedge y)$

$$\begin{aligned} U^j_{x,y} &= (x, y)^j \\ &= \textstyle\sum_{u \in \Omega_j} \iota(u)(x)\iota(u)(y) \\ &= \textstyle\sum_{u \in \Omega_j} [u \le x][u \le y] \\ &= \textstyle\sum_{u \in \Omega_j} [u \le x \wedge y] \\ &= \begin{cases} |\{u \in \Omega_j : u \le x \wedge y\}| & \text{if} \quad j \le l \\ 0 & \text{if} \quad j > l \end{cases} \\ &= \begin{bmatrix} l \\ j \end{bmatrix}_q A_{d-l}(x, y) \end{aligned}$$

QED.

Definition 5.4.
For $j = 0, 1, ..., d$*, let* π_j *be the orthogonal projection* $\pi_j : I\!R^X \to V_j$*. Then for each* $u \in \Omega_j$*, denote* $\check{u} = \pi_j(\iota(u))$*.*

Using the previous lemma, we obtain the following:

Corollary 5.5. *(of Lemma 5.3) For every* $j = 0, ..., d$*,* $\check{u}$ *is an eigenvector of* U^j *with eigenvalue* $\lambda_j = \sum_{l=j}^{d} \begin{bmatrix} l \\ j \end{bmatrix}_q p_{d-l}(j)$*, where* $p_i(j)$ *are the eigenvalues of* A_i *corresponding to the eigenspace* V_j*.*

Proof. Since $\check{u} \in V_j$ is an eigenvector of the adjacency matrices, the Corollary follows from the expression of U^j given in Lemma 5.3. QED.

Remark 5.6. Computation of λ_j.
From previous Corollary and making the change of variable $i = d - l$*, we have that* $\lambda_j = \sum_{i=0}^{d-j} \begin{bmatrix} d-i \\ j \end{bmatrix}_q p_i(j)$*.*

According to pages 261-265, 303-304 of ([2]);

$$p_i(j) = u_i(\theta_j)k_i \quad \text{where } k_i = p^0_{ii} \text{ and } u_i(\theta_j) \text{ is given by the basic hypergeometric series}$$

$$u_i(\theta_j) = {}_4\varphi_3\left(\begin{matrix} q^{-i}, & 0, & q^{-j}, & -q^{-d-e+j} & \\ & & & & ;q \quad ,q \\ q^{-d}, & 0, & 0, & & \end{matrix}\right) \text{defined by}$$

$$= \sum_{t=0}^{\infty} \frac{(q^{-i};q)_t\ (q^{-j};q)_t\ (-q^{-d-e+j};q)_t\ q^t}{(q^{-d};q)_t\ (q;q)_t} \quad \text{where}$$

$$(a;q)_t = \begin{cases} (1-a)...(1-aq^{t-1}) & (t=1,2,...) \\ 1 & (t=0) \end{cases}$$

From (3) *of page 1 and Theorem 9.4.3 of page 275 of* [1]

$$k_i = \begin{bmatrix} d \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie}$$

Caution: *The parameter "e" for dual polar graphs in page 303 of* [2]*, has been replaced by "e − 1" to follow the notation of* [1] *used in Lemma 4.2.*

Proposition 5.7.

$$\lambda_1 = q^{d-1} \prod_{j=-1}^{d-3} (1+q^{j+e}) = q^{d-1}(1+q^{e-1})\ a_2$$

(a_2 as in Lemma 4.3)

Proof. From Remark 5.6

$$\begin{aligned}
\lambda_1 &= \sum_{i=0}^{d-1} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q p_i(1) \\
&= \sum_{i=0}^{d-1} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q \left(\sum_{t=0}^{1} \frac{(q^{-i};q)_t\ (q^{-1};q)_t\ (-q^{-d-e+1};q)_t\ q^t}{(q^{-d};q)_t\ (q;q)_t} \right) \begin{bmatrix} d \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie} \\
&= \sum_{i=0}^{d-1} \left(1 + \frac{(1-q^{-i})(1-q^{-1})(1+q^{-d-e+1})\ q}{(1-q^{-d})(1-q)} \right) \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q \begin{bmatrix} d \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie}
\end{aligned}$$

since $\begin{bmatrix} i \\ 1 \end{bmatrix}_q = \frac{q^i-1}{q-1}$ and $\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{\begin{bmatrix} i \\ 1 \end{bmatrix}_q \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \cdots \begin{bmatrix} i-j+1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} j \\ 1 \end{bmatrix}_q \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q}$ we have $\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q \begin{bmatrix} d \\ i \end{bmatrix}_q = \begin{bmatrix} d \\ 1 \end{bmatrix}_q \begin{bmatrix} d-1 \\ i \end{bmatrix}_q$, thus

$$\begin{aligned}
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \sum_{i=0}^{d-1} \left(1 - \frac{\begin{bmatrix} i \\ 1 \end{bmatrix}_q (1+q^{d+e-1}) q^{1-e-i}}{\begin{bmatrix} d \\ 1 \end{bmatrix}_q} \right) \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie} \\
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \sum_{i=0}^{d-1} \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie} - \sum_{i=0}^{d-1} (1+q^{d+e-1}) q^{1-e-i} \begin{bmatrix} i \\ 1 \end{bmatrix}_q \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}+ie} \\
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \sum_{i=0}^{d-1} \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}} q^{ie} - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q (1+q^{d+e-1}) \sum_{i=1}^{d-1} \begin{bmatrix} d-2 \\ i-1 \end{bmatrix}_q q^{1-i} q^{\frac{(i^2-i)}{2}} q^{(i-1)e}
\end{aligned}$$

making change of variables $j = i-1$

$$\lambda_1 = \begin{bmatrix} d \\ 1 \end{bmatrix}_q \sum_{i=0}^{d-1} \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{\frac{(i^2-i)}{2}} q^{ie} - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q (1+q^{d+e-1}) \sum_{j=0}^{d-2} \begin{bmatrix} d-2 \\ j \end{bmatrix}_q q^{\frac{(j^2-j)}{2}} q^{je}$$

Using Newton's formula for Gaussian binomials

$$\prod_{k=0}^{n-1}(1+q^k t)=\sum_{k=0}^{n} q^{\frac{(k^2-k)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k$$

with $t=q^e$, we have

$$\begin{aligned}
&= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \prod_{j=0}^{d-2}(1+q^{j+e}) - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q (1+q^{d+e-1}) \prod_{j=0}^{d-3}(1+q^{j+e}) \\
&= \left((q^{d-1}+\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q)(1+q^{d-2+e}) - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q (1+q^{d+e-1}) \right) \prod_{j=0}^{d-3}(1+q^{j+e}) \\
&= q^{d-1}(1+q^{e-1}) \prod_{j=0}^{d-3}(1+q^{j+e}) \\
&= q^{d-1}(1+q^{e-1})\ a_2
\end{aligned}$$

QED.

Theorem 5.8. *For $j=0,...,d$, the set $\{\check{u}\}_{u\in\Omega_j}$ is a finite tight frame for V_j, i.e., there exists $c>0$ such that for all $f\in V_j$*

$$\sum_{u\in\Omega_j} <\check{u}, f>\check{u} = c\ f$$

Moreover, $c=\lambda_j$ is the eigenvalue of U^j corresponding to any $\check{u}$ with $u\in\Omega_j$.

Proof. From Corollary 5.5, we have that $\lambda_j\check{u}(x)=\sum_{y\in X}(x,y)^j\check{u}(y)$. Then, for an arbitrary $v\in\Omega_j$, we have:

$$\begin{aligned}
<\lambda_j\check{u},\check{v}> &= \sum_{x\in X}\lambda_j\check{u}(x)\check{v}(x) \\
&= \sum_{x\in X}\left(\sum_{y\in X}(x,y)^j\check{u}(y)\right)\check{v}(x) \\
&= \sum_{x,y\in X}\sum_{w\in\Omega_j}\iota(w)(x)\iota(w)(y)\check{u}(y)\check{v}(x) \\
&= \sum_{w\in\Omega_j}\left(\sum_{y\in X}\iota(w)(y)\check{u}(y)\right)\left(\sum_{x\in X}\iota(w)(x)\check{v}(x)\right) \\
&= \sum_{w\in\Omega_j}<\iota(w),\check{u}><\iota(w),\check{v}> \\
&= \sum_{w\in\Omega_j}<\check{w},\check{u}><\check{w},\check{v}> \qquad \text{(by orthogonality)} \\
&= <\sum_{w\in\Omega_j}<\check{w},\check{u}>\check{w},\check{v}>
\end{aligned}$$

Since this is true for an arbitrary $v\in\Omega_j$, we conclude that

$$\lambda_j\check{u}=\sum_{w\in\Omega_j}<\check{w},\check{u}>\check{w}$$

The statement of the theorem follows from the fact that $\{\check{w}\}_{w\in\Omega_j}$ span V_j. QED.

Lemma 5.9.
For all $\tau \in \Omega_1$, $\check{\tau} = \iota(\tau) - \frac{a_1}{|X|}\delta_X$ *with* a_1 *given in Lemma 4.3 and*
$\delta_X := \iota(\hat{0}) = 1 \in I\!R^X$

Proof. Recall that $< \iota(\hat{0}) >= \Lambda_0 \subseteq \Lambda_1 =< \{\iota(\tau)\}_{\tau\in\Omega_1} >$, and $\Lambda_1 = \Lambda_0 \oplus V_1$. Since $\check{\tau} = \pi_1(\iota(\tau)) \in V_1$, we have $\check{\tau} = \iota(\tau) - t.\delta_X$ for some $t \in I\!R$.
From the fact that $< \check{\tau}, \delta_X >= 0$ we conclude $t = \frac{<\iota(\tau),\delta_X>}{||\delta_X||^2} = \frac{\sum_{x\in X}[\tau\subseteq x]}{|X|} = \frac{a_1}{|X|}$.
QED.

Corollary 5.10. *Let* $h \in I\!R^X$, *then*

$$\pi_1(h) = \sum_{\tau\in\Omega_1} \frac{< \iota(\tau); h >}{\lambda_1}\check{\tau}$$

Proof. By Theorem 5.8, since $\pi_1(h) \in V_1$, we have

$$\pi_1(h) = \sum_{\tau\in\Omega_1} \frac{< \check{\tau}; \pi_1(h) >}{\lambda_1}\check{\tau}$$

but since $h = \pi_0(h) + \pi_1(h) + ... + \pi_d(h)$, $\pi_i(h) \in V_i$ and $< V_i, V_j >= 0\ \forall\ i \neq j$

$$\begin{aligned}
\pi_1(h) &= \sum_{\tau\in\Omega_1} \frac{< \check{\tau}; h >}{\lambda_1}\check{\tau}, \quad \text{by lemma 5.9} \\
&= \sum_{\tau\in\Omega_1} \frac{< \iota(\tau) - \frac{a_1}{|X|}\delta; h >}{\lambda_1}\check{\tau} \\
&= \sum_{\tau\in\Omega_1} \frac{< \iota(\tau); h >}{\lambda_1}\check{\tau} - \frac{< \frac{a_1}{|X|}\delta; h >}{\lambda_1} \sum_{\tau\in\Omega_1} \check{\tau} \\
&\qquad \text{but since } \sum_{\tau\in\Omega_1} \tau \in \Lambda_0 : \\
&= \sum_{\tau\in\Omega_1} \frac{< \iota(\tau); h >}{\lambda_1}\check{\tau}
\end{aligned}$$

QED.

6. Application: Norton product on V_1

Given the decomposition $I\!R^X = V_0 \oplus V_1 \oplus ... \oplus V_d$, in this section we describe the product of a Norton algebra attached to the eigenspace V_1.

Definition 6.1. *The Norton algebra on* V_1 *is the algebra given by the product* $f \star g = \pi_1(fg)$ *for* $f, g \in V_1$.

We want to compute the $\star$ product in V_1. Since $\Lambda_1 = span\{\iota(\tau) : \tau \in \Omega_1\}$ the set $\{\check{\tau}\}_{\tau\in\Omega_1}$ spans V_1.
We want to be able to compute $\check{\tau} \star \check{\sigma}$ in terms of this set of generators.

Lemma 6.2.

$$\check{\tau} \star \check{\sigma} = \pi_1(\iota(\tau \vee \sigma)) - \frac{a_1}{|X|}(\check{\tau} + \check{\sigma})$$

Proof. Recall that $\delta_X = 1 \in I\!R^X$ and that by Lemma 5.9 $\check{\tau} = \iota(\tau) - \frac{a_1}{|X|}\delta_X$.

Observe that δ_X is the identity of the product of functions. Then

$$\begin{aligned}
\check{\tau} \star \check{\sigma} &= (\iota(\tau) - \frac{a_1}{|X|}\delta_X) \star (\iota(\sigma) - \frac{a_1}{|X|}\delta_X) \\
&= \pi_1\big((\iota(\tau) - \frac{a_1}{|X|}\delta_X)(\iota(\sigma) - \frac{a_1}{|X|}\delta_X)\big) \\
&= \pi_1\big(\iota(\tau)\iota(\sigma) - \frac{a_1}{|X|}(\iota(\tau) + \iota(\sigma)) + (\frac{a_1}{|X|})^2\delta_X\big) \\
&= \pi_1\big(\iota(\tau)\iota(\sigma)\big) - \frac{a_1}{|X|}\pi_1(\iota(\tau) + \iota(\sigma)) + (\frac{a_1}{|X|})^2\pi_1(\delta_X) \quad \text{(by Lemma 4.6)} \\
&= \pi_1(\iota(\tau \vee \sigma)) - \frac{a_1}{|X|}(\check{\tau} + \check{\sigma})
\end{aligned}$$

QED.

It is clear that in order to complete the description of the product $\star$, we need to be able to calculate $\pi_1(\iota(\tau \vee \sigma)) = \pi_1(\iota(\tau)\iota(\sigma))$.

By Corollary 5.10:

$$\pi_1(\iota(\tau \vee \sigma)) = \sum_{\rho \in \Omega_1} \frac{< \iota(\rho);\ (\iota(\tau \vee \sigma)) >}{\lambda_1}\check{\rho}$$

Therefore we need to compute $< \iota(\rho); \iota(\tau \vee \sigma) >$. We do this in the following:

Lemma 6.3.

$$< \iota(\rho); \iota(\tau \vee \sigma) >= a_{\mathrm{rk}(\rho \vee \tau \vee \sigma)}$$

where a_j are as in Lemma 4.3.

Proof.

$$\begin{aligned}
< \iota(\rho); \iota(\tau \vee \sigma) > &= \sum_{x \in X} \iota(\rho)(x)\iota(\tau \vee \sigma)(x) \\
&= \sum_{x \in X} [\rho \leq x][\tau \vee \sigma \leq x] \\
&= \sum_{x \in X} [\rho \vee \tau \vee \sigma \leq x] \\
&= |\{x \in X : \rho \vee \tau \vee \sigma \leq x\}| \\
&= a_{\mathrm{rk}(\rho \vee \tau \vee \sigma)}
\end{aligned}$$

QED.

Definition 6.4. *Given $\tau, \sigma \in \Omega_1$, let:*

$$\Psi_j = \{\rho \in \Omega_1 : \mathrm{rk}(\rho \vee \tau \vee \sigma) = j\}$$

Theorem 6.5.

$$\check{\tau} \star \check{\sigma} + \frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) = \begin{cases} \check{\tau} & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \vee \sigma = \hat{1} \\ \frac{(1+q^{d-3+e})\sum_{\rho \in \Psi_2} \check{\rho} + \sum_{\rho \in \Psi_3} \check{\rho}}{q^{d-1}(1+q^{e-1})(1+q^{d-3+e})} & \text{otherwise} \end{cases}$$

Proof.

By Lemma 6.2, $\check{\tau} \star \check{\sigma} + \frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) = \pi_1(\iota(\tau \vee \sigma))$. Then, the cases $\tau = \sigma$ and $\tau \vee \sigma = \hat{1}$ follow.

In the remaining case, $\tau \vee \sigma \in \Omega_2$, so

$$\begin{aligned}\pi_1(\iota(\tau\vee\sigma)) &= \sum_{\rho\in\Omega_1}\frac{<\iota(\rho);\iota(\tau\vee\sigma)>}{\lambda_1}\check{\rho}\\ &= \sum_{\rho\in\Omega_1}\frac{a_{\mathrm{rk}(\rho\vee\tau\vee\sigma)}}{\lambda_1}\check{\rho}\quad(\diamond)\end{aligned}$$

Observe that $\mathrm{rk}(\rho\vee\tau\vee\sigma)\in\{2,3,d+1\}$. If $\mathrm{rk}(\rho\vee\tau\vee\sigma)=d+1, a_{d+1}=0$, otherwise $\rho\vee\tau\vee\sigma\in\Psi_2$ or Ψ_3.

Recall that $\lambda_1=q^{d-1}(1+q^{e-1})a_2$ and $a_2=(1+q^{e+d-3})a_3$ (Remark 4.4 and Proposition 5.7).

So $(\diamond)$ becomes

$$\begin{aligned}\pi_1(\iota(\tau\vee\sigma)) &= \frac{a_2\sum_{\rho\in\Psi_2}\check{\rho}+a_3\sum_{\rho\in\Psi_3}\check{\rho}}{\lambda_1}\\ &= \frac{(1+q^{e+d-3})\sum_{\rho\in\Psi_2}\check{\rho}+\sum_{\rho\in\Psi_3}\check{\rho}}{q^{d-1}(1+q^{e-1})(1+q^{d-3+e})}\end{aligned}$$

QED.

Remark 6.6. *Similar results are valid for the Johnson, Grassmann and Hamming cases. They are part of a current research.*

In the Johnson case $J(n,k)$, when $3\le k<\frac{n}{2}$, we obtain:

$$\check{\tau}\star\check{\sigma}=\begin{cases}(1-2\frac{k}{n})\check{\tau} & \text{if } \tau=\sigma\\ \frac{2k-n}{n(n-2)}(\check{\tau}+\check{\sigma}) & \text{if } \tau\neq\sigma\end{cases}$$

In the Hamming case, our formula for the Norton product reduces to zero. This is also direct from Theorem 5.2 of [4] *since $q^1_{1,1}=0$.*

In the Grassmann $J_q(n,k)$ case, when $3\le k<\frac{n}{2}$, we obtain:

$$\check{\tau}\star\check{\sigma}+\frac{q^k-1}{q^n-1}(\check{\tau}+\check{\sigma})=\begin{cases}\check{\tau} & \text{if } \tau=\sigma\\ \frac{q^{k-1}-1}{q(q^{n-2}-1)}\sum_{\rho\in\Psi_2}\check{\rho} & \text{if } \tau\neq\sigma\end{cases}$$

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